16 Composition of Functions

In this investigation, we will be **iterating** functions of the form $f(x) = \frac{B}{D-x}$. In other words, we will be composing them with themselves. We will start by doing some numerical experimentation, which will give us an idea of how the iteration works. We will do some algebra composing these functions and note how they form groups, which are isomorphic to some of the symmetry groups we've investigated. Then, we will see how these functions represent geometric transformations, which can in turn be composed and decomposed. Finally, we will find sets of matrices that behave in the same way under multiplication as these functions do under composition. By the time our investigation is through, we should have reviewed many of the themes that we became acquainted with in this book!

Some review: $f^n(x)$ means iterating f on itself n times, or $\underbrace{f(f(\cdots f(x) \cdots))}_{n \text{ times}}$. $\circ$ means to compose functions. For example, $f \circ f \circ f = f^3$.

To begin with, let's look at the function $f(x) = \frac{1}{1-x}$. We will want to find $f(f(x)) = f^2(x)$, $f^3(x)$, etc. Rather than doing the algebra, let's just plug in $x = 5$. To do this efficiently:

1. Enter the number 5 into your calculator.

2. On the next line, type $1/(1 - \text{Ans})$.

3. Hit the enter key a few times; you will find that $f(5) = -.25$, $f^2(5) = .8$, $f^3(5) = 5$, etc.

Clearly, this function has a period of 3 when the initial value is 5.

4. Try a few other values for x to see if and how the periodicity of f depends on x.

5. Look at your results of iterating the following functions, and make a table of your results. If the initial value has a problem, record that.

	$\frac{1}{1-x}$	$\frac{2}{2-x}$	$\frac{3}{3-x}$	$\frac{4}{4-x}$	$\frac{5}{5-x}$	$\frac{7}{7-x}$	$\frac{6}{1-x}$	$\frac{6}{2-x}$	$\frac{6}{3-x}$	$\frac{9}{3-x}$
init.	5	5	5	5	5	5	5	5	5	5
$f(x)$	$-.25$									
$f^2(x)$	$.8$									
$f^3(x)$	5									
$f^4(x)$	$-.25$									
$f^5(x)$	$.8$									
$f^6(x)$	5									
$f^7(x)$	$-.25$									
$f^8(x)$										

6. (a) At this point, you should have noticed some fundamentally different types of behavior of $f(x) = \frac{B}{D-x}$ under iteration, depending on the values of B and D. What are they?

   (b) Do you have any theories yet as to when to expect each type of behavior for given values of B and D?

We'll try a slightly more algebraic approach to the problem. Composing $f(x) = \frac{1}{1-x}$ with itself yields:

$$f^2(x) = \frac{1}{1 - \frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x}$$

$$f^3(x) = \frac{1}{1 - \frac{x-1}{x}} = \frac{x}{x - (x-1)} = x.$$

Let's tabulate these results as a group next to the rotation group of the equilateral triangle. Indeed, they are the same, as shown in Figure 1. Note that r is a rotation of $120°$.

For Problems 7–9,

   i. compose each until x is reached;

   ii. record each of the functions generated;

   iii. produce a group table;

   iv. identify a symmetry group to which each group is isomorphic.

∘	x	f	f^2		$\cdot$	I	r	r^2
x	x	f	f^2		I	I	r	r^2
x	f	f^2	x		r	r	r^2	I
f^2	f^2	x	f		r^2	r^2	I	r

Figure 1: The group generated by $f(x) = \frac{1}{1-x}$, next to C_3.

7. $f(x) = \frac{2}{2-x}$ 8. $f(x) = \frac{3}{3-x}$ 9. $f(x) = \frac{9}{3-x}$

10. The function $f(x) = \frac{1}{2-x}$ behaves differently under composition.

    (a) Make a record of the first few compositions of $f(x)$ with itself.

    (b) Write a formula for $f^n(x)$.

    (c) Write a formula for its inverse function, $f^{-1}(x)$.

    (d) Why is $f^{-1} \circ f^{-1} = f^{-2}$?

    (e) What is $f^{-2} \circ f^2$?

    (f) Taken together, $f(x)$ and $f^{-1}(x)$ generate a group under composition. How big is it?

    (g) Substitute a value for x and iterate. What happens?

11. The function $f(x) = \frac{2}{1-x}$ illustrates another behavior under composition.

    (a) Plug in a value for x and use your calculator to see what happens.

    (b) Find the inverse of this function.

    (c) The original function and its inverse once again generate a group under composition. How big is this group?

    (d) Is it isomorphic to the group in Problem 7?

    (e) If you have time, write out a few of the elements of the group. Do you see any pattern?

12. Looking back at your results to Problems 4–7, you will notice that the elements of each group are not all in the original form $f(x) = \frac{B}{D-x}$. All, however, are in the form $f(x) = \frac{Ax+B}{Cx+D}$. If $f(x) = \frac{Ax+B}{Cx+D}$ for some A, B, C, D, can you always write f^n in the form $f^n(x) = \frac{Px+Q}{Rx+S}$ for some P, Q, R, S?

We will temporarily narrow our focus to the class of functions $f(x) = \frac{1}{D-x}$. We will use algebra to see how D affects the periodicity of f.

First, we solve for D when f has period 1:

$$f(x) = \frac{1}{D-x} = x$$
$$D = \frac{x^2+1}{x}.$$

So there is no constant value of D which gives f a period of 1. We'll try out periods 2 and 4 later, but let's see 3 and 5 first. We set $f^3(x) = x$ and proceed:

$$f^3(x) = f(f(f(x))) = f\left(f\left(\frac{1}{D-x}\right)\right)$$
$$= f\left(\frac{1}{D - \frac{1}{D-x}}\right)$$
$$= f\left(\frac{D-x}{D^2 - Dx - 1}\right)$$
$$= \frac{1}{D - \frac{D-x}{D^2-Dx-1}} = x$$
$$\implies D^2 - Dx - 1 = D^3x - D^2x^2 - 2Dx + x^2$$

$$\implies D^3x - D^2(x^2 + 1) - Dx + (x^2 + 1) = 0$$
$$\implies (D^2 - 1)(Dx - (x^2 + 1)) = 0.$$

Setting $D^2 - 1 = 0 \to D = \pm 1$ produces a function with period 3. That is, $f(x) = \frac{1}{1-x}$ and $f(x) = \frac{1}{-1-x}$ have period 3 under composition. You should recognize one of these already.

Looking at period 5, we set $f^5(x) = x$. Some nasty algebra is omitted.

$$f^5(x) = f(f(f^3(x))) = f\left(f\left(\frac{1}{D - \frac{D-x}{D^2 - Dx - 1}}\right)\right)$$
$$= \frac{D^4 - D^3x - 3D^2 + 2Dx + 1}{D^5 - D^4x - 4D^3 + 3D^2x + 3D - x} = x$$
$$\implies (Dx - (x^2 + 1))(D^4 - 3D^2 + 1) = 0$$
$$\implies D^2 = \frac{3 \pm \sqrt{5}}{2}$$
$$\implies D = \pm\sqrt{\frac{3 \pm \sqrt{5}}{2}}$$

So we have that $f(x) = \frac{1}{\pm\sqrt{\frac{3\pm\sqrt{5}}{2}} - x}$ for period 5.

This algebra is getting annoying, but matrices can simplify the process. In particular, we can make a correspondence between the functions generated by composing functions of the form $f(x) = \frac{Ax+B}{Cx+D}$ and the matrices generated by multiplying matrices of the form $F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

13. (a) Let $f(x) = \frac{Ax+B}{Cx+D}$, $g(x) = \frac{Px+Q}{Rx+S}$, and $h(x) = \frac{Tx+U}{Vx+W}$. Show that if $f \circ g = h$, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} T & U \\ V & W \end{bmatrix}.$$

Do this by computing $f(g(x))$ and the matrix product, then comparing.

(b) This is not a true one-to-one correspondence. Why not? (Hint: think about when division is undefined, as compared to matrix multiplication.)

We will now use this "isomorphism" to make the job of composing functions easier. If $f(x) = \frac{1}{D-x}$, then $F = \begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}$ corresponds to f.

$$F^2 = \begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}$$
$$= \begin{bmatrix} -1 & D \\ -D & D^2 - 1 \end{bmatrix}$$
$$\implies \boxed{f^2(x) = \frac{D-x}{D^2 - Dx - 1}}$$
$$F^3 = FF^2 = \begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}\begin{bmatrix} -1 & D \\ -D & D^2 - 1 \end{bmatrix}$$
$$= \begin{bmatrix} -D & D^2 - 1 \\ 1 - D^2 & D^3 - 3D \end{bmatrix}$$
$$\implies \boxed{f^3(x) = \frac{(D^2 - 1) - Dx}{(1 - D^2)x + (D^3 - 2D)}}$$

14. Find $f^4(x)$, $f^5(x)$, and $f^6(x)$ using matrices.

Recall that to find what values of D result in period n, we solve $f^n(x) = x$. Now, you can get to solve for values of D, which will produce periods of two, four, and six. Some things to consider:

- We had no solutions for period one, two solutions for period three, and four solutions for period five. How many do you expect for periods two, four, and six?

- If a function repeats itself every two iterations, will it necessarily repeat itself every four iterations? Every six iterations?

- Given the previous question, what do we know about some of the solutions for D for periods four and six, as compared with the true periods?

15. Use algebra or a computer algebra system—if you're lazy—to solve the following equations for D. Use f^2 from my writing above; use f^4 and f^6 from your answers to Problem 11.

    (a) $f^2(x) = x$ (b) $f^4(x) = x$ (c) $f^6(x) = x$

16. For each period, choose one of your solutions for D, then use your calculator to iterate $f(x) = \frac{1}{D-x}$.

    (a) Period 2:   (b) Period 4:   (c) Period 6:

$$D = \underline{\quad}$$
$$f(x) = \underline{\quad}$$
Found period $= \underline{\quad}$

$$D = \underline{\quad}$$
$$f(x) = \underline{\quad}$$
Found period $= \underline{\quad}$

$$D = \underline{\quad}$$
$$f(x) = \underline{\quad}$$
Found period $= \underline{\quad}$

Now, we turn our attention to fixed points. These are points where $f(x) = x$. We already discovered that there is no constant D such that $\frac{1}{D-x} = x$ for *all* x, but it is true for *some* x. In particular, you can solve the equation to find $x = \frac{D \pm \sqrt{D^2-4}}{2}$. Use this fact in the following problem:

17.   (a) For what values of D is x real?

    (b) For what values does x have a nonzero imaginary component?

    (c) What is true of the fixed points for all of the cases where $f(x)$ is periodic?

It looks like the natural home of the function $f(x) = \frac{1}{D-x}$ is actually in the complex plane. Let's rename it to $f(z) = \frac{1}{D-z}$, since we're dealing with complex numbers now. Thus, $z = \frac{D \pm \sqrt{D^2-4}}{2}$. We'll continue to say that D is real, though.

18.   (a) Plot a graph in the complex plane of all the fixed points as D ranges over the real numbers. Your graph should be a circle and two rays.

    (b) Plot and label the fixed points associated with D values for periods $2, 3, 4, 5$. What do you notice?

    (c) Make a prediction about the values of the fixed points associated with functions of period 7 and 8.

    (d) Then predict what values of D will produce periods of 7 and 8 for $f(z) = \frac{1}{D-z}$ under iteration.

    (e) Use your calculator to iterate your functions for $7, 8$ and confirm your hypothesis.

    (f) Use your calculator to show that the fixed points for $7, 8$ remains fixed.

    (g) Where do you suppose you would locate the fixed point so that you had an "infinite" period?

    (h) Construct an example of $f(z) = \frac{1}{D-z}$ with $|D| < 2$ and an infinite period.

    (i) Once again, check using your calculator.

19.   (a) Find the fixed points for:

    i. $f(z) = \frac{1}{2-z}$ ii. $f(z) = \frac{1}{3-z}$ iii. $f(z) = \frac{1}{4-z}$

    (b) Use your calculator to iterate these functions using a variety of initial values.

    (c) What do you notice about these functions?

By this time, you should have a good sense of the behavior of $f(z) = \frac{1}{D-z}$ under iteration. You should know how to select the value of D so that the function approaches a fixed point, has a finite period of n, or neither. The reasoning behind why this is, however, is likely still unclear.

To begin to answer these questions, we will investigate our function from a geometric, rather than algebraic, point of view. Until this point, we have been composing our function. Decomposing it, however, will shed more light on the subject.

We willl begin by decomposing $f(z) = \frac{1}{1-z}$. In doing so, we must find two functions $g(z)$ and $h(z)$ with $g(h(z)) = f(z)$. The most obvious way to do this is to let $g(z) = \frac{1}{z}$ and $h(z) = 1 - z$, but this will result in some obscure geometry.

The more useful way to decompose it is to let $g(z) = \frac{1}{\overline{z}}$ and $h(z) = \overline{1 - z}$.[16] If $z = a + bi$ for real a and b, then by definition, $\overline{z} = a - bi$. So we have

$$g(h(z)) = g(\overline{1 - z}) = \overline{\left(\frac{1}{\overline{1 - z}}\right)} = \frac{1}{1 - z} = f(z).$$

20. Let's look at the geometry of g and h by completing the table in Figure 3. After doing this, you should be able to characterize how $g(z)$ and $h(z)$ transform the complex plane.

21. Find the set of points which remain fixed under $h(z)$.

    (a) What is an equation for it?

    (b) Give it a geometric name. It will help to complete the table first.

22. Find the set of points which remain fixed under $g(z)$.

    (a) What is an equation for it?

    (b) Give it a geometric name.

23. There are two points which remain fixed under both $g(z)$ and $h(z)$.

    (a) What are they?

    (b) Would they remain fixed under $g(h(z))$? Why or why not?

24. What transformation of the plane does $h(z)$ produce?

25. What transformation of the plane does $g(z)$ produce?

The transformation produced by $h(z)$ is a reflection over the line $x = .5$. The transformation produced by $g(z)$ is called an **inversion** over the unit circle. It is similar to a reflection, and if the preimage is very close to the unit circle the image is almost identical to the reflection of the preimage over the circle's circumference. The two transformations are depicted for various points in Figures 2 and 4.

26. Recall that successive reflections over two intersecting lines produce a rotation around the point of intersection, by twice the angle between the two lines.

    (a) What is the angle between the unit circle and the line $x = .5$ at the point of intersection?

    (b) If you chose your initial value of x to be "close to" one of the fixed points, what would you expect to happen if you transform it with $g(h(z))$...

    i. ... once?   ii. ... twice?   iii. ... thrice?

27. (a) Use your calculator to test your hypothesis by picking a point close to $.5(1 + \sqrt{3})$, like $.5 + .85i$, as an initial value. Then, iterate $f(x) = \frac{1}{1-z}$.

    (b) What happens as you move your initial value farther from the fixed point?

28. It appears that this composition of functions produces a rotation-like transformation, but there are two "competing" centers with the nearer center to a point "exerting more influence." The "rotation" is distorted more and more the farther you are from the fixed points. What happens when the initial value is on the real axis?

29. What is the period of

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[16]Remember that $\overline{z}$ is the complex conjugate.

z	$h(z) = \overline{1-z}$	z	$g(z) = \frac{1}{z}$
i	$1+i$	i	i
0		0	
$.5$		$.5$	
2		2	
3		3	
-1		-1	
-2		-2	
-3		-3	
$2i$		$2i$	
$3i$		$3i$	
$-i$		$-i$	
$1+i$		$1+i$	
$.5+i$		$.5+i$	
$.5+2i$		$.5(1+i)$	
$.5-i$		$.5(-1+i\sqrt{3})$	
$-.5+i$		$.5(-1-i\sqrt{3})$	
$.5(1+i\sqrt{3})$		$.5(1+i\sqrt{3})$	
		$.5(1-\sqrt{3})$	
		$.5(i+\sqrt{3})$	
		$.5(\sqrt{2}+i\sqrt{2})$	

Figure 2: The transformation $h(z) = \overline{1-z}$.
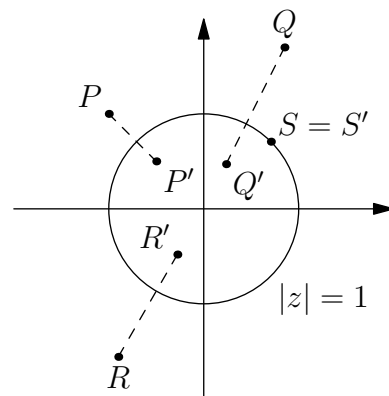
Figure 4: The transformation $g(z) = \frac{1}{z}$.

Figure 3: Some values to try for z for g and h.

(a) $g(z) = \frac{1}{z}$?

(b) $h(z) = \overline{1-z}$?

We will now return to our original idea for decomposing $f(z) = \frac{1}{1-z}$ as $g(z) = \frac{1}{z}$ and $h(z) = 1-z$. We will take an algebraic point of view, so it makes it easier to not use conjugation.

30. We will compose $g(z) = \frac{1}{z}$ and $h(z) = 1-z$. Feel free to use a computer algebra system to do these problems.

(a) $g \circ g$

(b) $h \circ h$

(c) $g \circ h$

(d) $g \circ g \circ h$

(e) $g \circ h \circ h$

(f) $g \circ g \circ h$

(g) $g \circ h \circ g$

(h) $g \circ h \circ g \circ h$

(i) $g \circ h \circ g \circ h \circ g$

(j) $g \circ h \circ g \circ h \circ g \circ h$

(k) $h \circ g$

(l) $h \circ g \circ h$

(m) $h \circ g \circ h \circ g$

(n) $h \circ g \circ h \circ g \circ h$

(o) $h \circ g \circ h \circ g \circ h \circ g$

31. Let $j(z) = g \circ h \circ g(z)$ and let $k(z) = g \circ h \circ g \circ h(z) = f^2(z)$. Also, let $I(z) = z$ be the identity function. In the group table, let the function on the top be in the inner function, and the function on the left be the outer function, so for example $g \circ h(z) = f(z)$ as shown.

(a) Complete the group table.

(b) Is this group commutative?

(c) What group(s) is it isomorphic to?

(d) What subgroup(s) does it have?

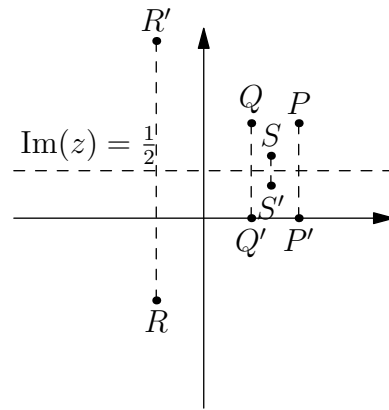| $\circ$	I	f	k	g	j	h
I						
f						
k						
g						f
j						
h						

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32. Repeat the above analysis on the function (i) $f(z) = -\frac{1}{z}$, (ii) $f(z) = \frac{1}{\sqrt{2}-z}$, and (iii) $f(z) = \frac{1}{\sqrt{3}-z}$.

    (a) Decompose it into two functions involving complex conjugates and graph the fixed points of each function.

    (b) Find the intersection of the two sets of fixed points.

    (c) Describe what each of the functions does.

    (d) Decompose the function into two functions that do not involve conjugates and recompose them.

    (e) Make a group table.

    (f) What groups is this group isomorphic to?

33. Now, consider the function $f(z) = \frac{1}{2-z}$.

    (a) Decompose it into two functions involving complex conjugates and graph the fixed points of each function.

    (b) Where do they intersect?

    (c) Use your graph to visualize what happens when you take a point, reflect it over the line, and invert it about the circle.

    (d) Iterate this process—don't use your calculator for now. Where do you end up?

    (e) Use your calculator to confirm your prediction, keeping track of where you end up on the graph after each iteration.

    (f) Use a variety of initial points, and don't only use real ones.

34. Repeat the process of the previous problem for the functions $f(z) = \frac{1}{3-z}$ and $f(z) = \frac{1}{4-z}$.

    (a) Decompose them using conjugates, and graph the set of fixed points.

    (b) Note that although the circle of inversion and reflection line do not intersect, $f(x)$ still has two fixed points. Find them algebraically.

    (c) Note that the two fixed points play a fundamentally different role. one is called an attractive fixed point, while the other is a repelling fixed point. Why?

    (d) Was this the case for D values which produced non-real fixed points?

    (e) How does iterating functions like $f(z) = \frac{1}{3-z}$ compare to repeated, sequential reflection over two parallel lines?

By now, you should have a good geometric sense of what causes the function $f(x) = \frac{1}{D-x}$ to oscillate when $|D| < 2$, to be periodic with a period of n when $|D| < 2$ and $D = 2\cos\frac{k\pi}{n}$ for integers k, n, and to approach a fixed point when $|D| \geq 2$.

    We would still like to make sure that our *geometric* insights are *algebraically* correct. To do that, we will take another look at the correspondence

$$f(x) = \frac{Ax + B}{Cx + D} \iff F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Consider the matrix $\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}$. Under multiplication, it will be periodic with period n if and only if

$$\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recall that when a matrix M acts on a vector v, it usually changes its directions, but that if it has independent eigenvectors v_1 and v_2, they will act as though they are being multiplied by a scalar constant. That is, $Mv_1 = \lambda_1 v_1$ and $Mv_2 = \lambda_2 v_2$ for some complex λ_1, λ_2. Also, recall that any vector in the plane can be decomposed as $v = av_1 + bv_2$. Because M is periodic, $M^n v_1 = I v_1$, but $M^n v_1 = \lambda_1^n v_1$, so we need to find which values of D produces eigenvalues where $\lambda_1^n = 1$: the roots of unity.

35. Solve the eigenvalues λ_1 and λ_2 of $\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}$. In the course of doing so, you should come across an equation which reminds you of the formula for fixed points.

36. For what values of D do the eigenvalues have magnitudes of 1? For what values of D are they n^{th} roots of unity? Express your answer as a trig function of n.

37. Let $D = 1$, so $F = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

    (a) What are the eigenvalues of F? Express them in terms of $\operatorname{cis} \theta$.

    (b) What is the period of this matrix under multiplication with itself?

    (c) Explain the apparent discrepancy between the period of this matrix under multiplication with itself and that of $f(z) = \frac{1}{1-z}$ under iteration.

38. Look at your answers to $f(z) = \frac{2}{2-z}$ and $f(z) = \frac{3}{3-z}$ from the first table in this chapter.

    (a) What were their periods?

    (b) What is the circle of inversion for each one?

    (c) Write the matrices which correspond to $f(z) = \frac{2}{2-z}$ and $f(z) = \frac{3}{3-z}$.

    (d) Are the determinants of these matrices 1?

    (e) Will these matrices ever repeat themselves under successive multiplication?

    (f) Find the eigenvalues of these matrices.

    (g) Are the eigenvalues' magnitudes 1?